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## Derivation of low-temperature expansions for Ising model VII. The honeycomb-triangular code system

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**Abstract.** The honeycomb-triangular code system is studied. It is shown how the principle of complete code balance can be exploited in a systematic way and the method is generalized to apply to partial codes. Euler's law of the edges is used to establish a latent symmetry property of the code system. The new results make possible the derivation of extended series.

### 1. Introduction

In this paper we continue our study of the derivation of series expansions for the triangular and honeycomb lattices both as a field or  $\mu$ -grouping and as a temperature or  $u$ -grouping. We have introduced the problem and given the elementary general theory in previous papers (Sykes *et al* 1965, 1973a, b, c, to be referred to as I, II, III, IV respectively).

Apart from the intrinsic interest of new data for the study of the simple Ising model the detailed configurational information obtained at the same time has application to many other problems that arise in the theory of cooperative phenomena. Notably multi-spin models (Sykes and Watts 1975 and references there cited), general spin models (Sykes and Gaunt 1973 and references there cited), order-disorder transitions in non-stoichiometric binary alloys and the staggered susceptibility of the Ising antiferromagnet (see Sykes *et al* 1973d for a lead into the literature), the derivation of low- and high-density expansions for hard-sphere lattice gases (Gaunt and Fisher 1965, Gaunt 1967) and configurational studies of Potts models (Domb 1974, Ditzian 1974, Straley and Fisher 1973, Enting 1974a, b).

In III the field grouping for the triangular lattice was obtained to order ten, taking account of all the perturbations of up to ten spins; even at this order the enumeration problem is quite difficult. We describe how the principle of complete code balance (II, §2) can be exploited in a systematic way. A subset of codes (secondary codes) can be calculated recursively from the complementary set of (primary) codes. The method can be generalized to apply to the temperature grouping for which we describe a *principle of partial code balance*.

The honeycomb-triangular code system is found to possess a property of *latent symmetry* which can be established by an application of Euler's law of the edges to the underlying shadow graphs. By exploiting this latent symmetry, in conjunction with the principle of balance, five new ferromagnetic polynomials  $\psi_{17}$  through  $\psi_{21}$  have been added to the temperature grouping for both the triangular and honeycomb lattices.

For the triangular lattice two new field polynomials  $L_{11}$  and  $L_{12}$  have been obtained and from these and the corresponding complete codes  $F_{11}$  and  $F_{12}$  four new field polynomials  $L_{22}$  through  $L_{25}$  for the honeycomb lattice. Companion papers (Sykes *et al* 1975a, b, to be referred to as VIII and IX) describe the detailed application of the techniques and give the new results.

**2. Principle of complete code balance**

The principle of complete code balance is stated in detail in II, § 2; it implies a set of constraints on each complete code  $F_n$  which must be consistent with all the lower-order complete codes  $F_0$  through  $F_{n-1}$ . We now show that for the honeycomb-triangular code system these constraints can be used to determine all the codes  $(\lambda, \alpha, \beta, \gamma)$  in  $F_n$  with  $\beta < \alpha$  if all the codes with  $\beta \geq \alpha$  are known. For convenience we call the codes characterized by  $\beta \geq \alpha$  primary codes and those characterized by  $\beta < \alpha$  secondary codes. We further show that the secondary codes can be calculated recursively from the primary codes in a systematic way.

By definition (II, equation (2.9)) the value of  $\gamma$  is the class of a code. Suppose that in  $F_n$  all codes with class greater than  $C$  have been determined. There are  $n - C$  codes of class  $C$  with  $\beta < \alpha$ ; explicitly:

$$\begin{aligned}
 C_0 &= (3n - 2C, 3n - 3C, 0, C) \\
 C_1 &= (3n - 2C - 1, 3n - 3C - 2, 1, C) \\
 &\vdots \\
 C_j &= (3n - 2C - j, 3n - 3C - 2j, j, C) \\
 &\vdots \\
 C_{n-C-1} &= (2n - C + 1, n - C + 2, n - C - 1, C).
 \end{aligned}
 \tag{2.1}$$

Now the codes in  $F_n$  are interpreted on the honeycomb lattice by the substitution (II, equation (2.6)):

$$(\lambda, \alpha, \beta, \gamma) = Y^n(1 + bX)^\alpha(1 + b^2X)^\beta(1 + b^3X)^\gamma(1 + X)^{-\lambda}
 \tag{2.2}$$

and the coefficients of the sequence

$$Y^n X^C b^{3C}, Y^n X^{C+1} b^{3C+2}, \dots, Y^n X^{C+j} b^{3C+2j}, \dots, Y^n X^{n-1} b^{3C+2n-2C-2}
 \tag{2.3}$$

which we call the *class C balance sequence* must reproduce exactly those of the corresponding sequence, with  $X$  and  $Y$  interchanged, generated by the codes  $F_C, \dots, F_{n-1}$ . We have chosen the sequence (2.3) because, as can be verified by inspection, no code of class less than  $C$  can contribute to any coefficient; but we have supposed codes of class greater than  $C$  determined; the principle of complete code balance, which predetermines the  $n - C$  coefficients of (2.3) therefore implies  $n - C$  constraints on the codes of class  $C$ . If all the primary codes ( $\beta \geq \alpha$ ) of class  $C$  are known these constraints are sufficient to determine the remaining  $n - C$  class  $C$  codes. Further the only codes of class  $C$  which contribute to the coefficient of  $Y^n X^{C+j} b^{3C+2j}$  form an ascending sequence  $C_j, C_{j+1}, \dots$  in (2.1) and so the resulting equations can be solved recursively, starting with the coefficient of  $C_{n-C-1}$ . *By repeating this process systematically in descending order of class all the  $\frac{1}{2}n(n+1)$  (secondary) codes in  $F_n$  with  $\beta < \alpha$  can be determined if all the (primary) codes with  $\beta \geq \alpha$  are known.*

A practical application of the above procedure has been made by Betts *et al* (1974) to the hydrogen peroxide lattice. The code system there studied may be regarded as equivalent to the honeycomb-triangular system so far as the present properties are concerned. For other code systems in which the class of a code is determined by more than one parameter of the code the balance sequences yield a set of independent implicit equations which cannot in general be solved recursively.

### 3. Principle of partial balance

In practical applications it often arises that the primary codes in  $F_n$  are known only if they correspond to some power,  $m$  say, of  $u$  or less. If we denote the power of  $u$  on the triangular lattice by  $\omega$  then we have the relation (from II, equation (3.5))  $m \geq \omega = \alpha + \beta$ . In other words the primary codes are known only if they lie in the *partial code*  $F_n^m$  (as defined previously in IV, § 2, this denotes the code corresponding to all graphs on the triangular lattice with  $\omega \leq m$ ).

It can be shown that if all the primary codes in  $F_n^m$  are known the procedure described for complete codes can be used without modification and all the secondary codes in  $F_n^m$  can still be found recursively using the balance sequences generated by the lower-order complete codes  $F_C \dots F_{n-1}$ . It can be further shown that these sequences are correctly generated by the *partial codes*  $F_C^m, \dots, F_{n-2}^m, F_{n-1}^m$ . These results, which we call collectively the *principle of partial balance*, are included by implication in the following theorem.

#### Theorem

Suppose  $(\omega + C, -3n + 3C + 2\omega, 3n - 3C - \omega, C)$  is any code in  $F_n$  for which  $\omega > 2n - 2C$  and which is therefore a secondary code determined by the coefficient of  $Y^n X^n b^{C+2n'}$  with  $n' = 3n - 2C - \omega$  in the class  $C$  balance sequence. Then any code  $(\omega' + C', -3n' + 3C' + 2\omega', 3n' - 3C' - \omega', C')$  with  $n' \leq n$  contributes to the coefficient of  $Y^n X^n b^{C+2n'}$  if, and only if,

$$C' \geq C - 2(n - n') \tag{3.1}$$

$$\omega' \leq \omega - (C' - C) + 2(n' - n) \tag{3.2}$$

with the trivial restriction  $\omega' \geq 3$ .

We do not give a detailed proof of this theorem; it may be established by an exhaustive examination of all the possible cases that can arise in picking out the required coefficient in the expansion of the right-hand side of (2.2).

The burden of the principle of partial balance is that all the secondary codes in the temperature grouping are determined by all the primary codes in the temperature grouping; further the secondary codes can always be calculated recursively, using the receipt given in § 2.

### 4. Latent symmetry of the honeycomb-triangular code system

In this section we describe a special symmetry property of the honeycomb-triangular code system which depends essentially on the symmetry between significant and insignificant triangles (defined in II, § 3). We first express the general code in terms of the graph

parameters of the corresponding shadow graph. Any code corresponds to one or more shadow graphs and each shadow graph is a configuration of perturbed spins which can be described by a linear graph. We summarize our notation for the parameters which is consistent with I–IV:

- $s$  = number of spins (vertices of the graph)
- $r$  = number of first neighbour bonds (edges of the graph)
- $c$  = number of connected components
- $f$  = number of finite faces
- $t$  = number of elementary triangular faces
- $\xi$  = number of elementary triangular faces of significant parity
- $\eta$  = number of elementary triangular faces of insignificant parity
- $h$  = number of *holes* defined as  $f - t$
- $\omega$  = corresponding power of  $u$  on triangular lattice.

Clearly

$$f = h + t = h + \xi + \eta \quad (4.1)$$

and we recall that (II, equation (3.4))

$$\omega = \alpha + \beta = 3s - r. \quad (4.2)$$

Since all the configurations are *planar* graphs we have the result, usually called Euler's law of the edges, which in our notation may be written

$$f = r - s + c. \quad (4.3)$$

It follows by eliminating  $f$  using (4.1) and (4.2) that

$$t = \xi + \eta = 2s - \omega + c - h. \quad (4.4)$$

All configurations that contribute to the coefficient of  $u^\omega \mu^s$  on the triangular lattice can be divided into subsets characterized by their value of  $t$ , which value is determined by the quantity

$$\kappa = c - h \quad (4.5)$$

which we call the *discriminant* of the graph. (The same discriminant arises in the theory of the Ising model with pure triplet interactions (Sykes and Watts 1975).)

If  $\omega$  and  $s$  are fixed any code of class  $C$  is uniquely determined; further the class  $C$  is just the number of triangles of significant parity ( $\xi$ ) in the corresponding configuration. Because of the symmetry between triangles of significant and insignificant parity on the lattice, every configuration with a distribution of  $t$  into  $\xi$  and  $\eta$  will imply the existence of another with the roles of  $\xi$  and  $\eta$  reversed; this will ensure a symmetric pattern in the coefficients of the codes for any fixed  $s$ ,  $\omega$  and  $\kappa$  with centre at

$$\xi = \eta = \frac{1}{2}(2s - \omega + \kappa). \quad (4.6)$$

We expect one or two central codes depending on whether  $t$  is even or odd. We write

$$t = 2s - \omega + c - h = \begin{cases} 2l & t \text{ even} \\ 2m - 1 & t \text{ odd} \end{cases} \quad (4.7)$$

and it is convenient to consider separately the odd regime, corresponding to  $\omega$  odd, and the even regime, corresponding to  $\omega$  even.

The corresponding codes are explicitly:

Odd regime

$$\begin{aligned}
 & \vdots \\
 & (2s-l+\kappa+2, s-l+2\kappa+6, s-l-\kappa-6, l+2) \\
 & (2s-l+\kappa+1, s-l+2\kappa+3, s-l-\kappa-3, l+1) \\
 & * (2s-l+\kappa, s-l+2\kappa, s-l-\kappa, l) \\
 & (2s-l+\kappa-1, s-l+2\kappa-3, s-l-\kappa+3, l-1) \\
 & (2s-l+\kappa-2, s-l+2\kappa-6, s-l-\kappa+6, l-2) \\
 & \vdots
 \end{aligned}$$

Even regime

(4.8)

$$\begin{aligned}
 & \vdots \\
 & (2s-m+\kappa+2, s-m+2\kappa+5, s-m+\kappa-4, m+1) \\
 & (2s-m+\kappa+1, s-m+2\kappa+2, s-m-\kappa-1, m) \\
 & * (2s-m+\kappa, s-m+2\kappa-1, s-m-\kappa+2, m-1) \\
 & (2s-m+\kappa-1, s-m+2\kappa-4, s-m-\kappa+5, m-2) \\
 & \vdots
 \end{aligned}$$

where the asterisk marks the centre of symmetry in the coefficients.

In general the symmetry of each subset of codes characterized by the values of  $l$  and  $\kappa$  will be *latent*, since it will be hidden by the overlapping of symmetric code patterns resulting from a range of values of the discriminant  $\kappa$ . If all the configurations are connected graphs without holes the code patterns (4.8) will exhibit a *patent* symmetry corresponding to  $\kappa = 1$ . In this latter case, for both the even and odd regimes, the codes on or above the centre of symmetry satisfy the condition  $\beta < \alpha$  (and are therefore *secondary* codes); it follows from the results of §§ 2 and 3 that they are determined by the principle of partial balance. The inequalities (3.1) and (3.2) ensure that for any fixed  $\omega$  the primary codes are not used in the recursion. But all these may be completed by the symmetry of the system (4.8) and therefore any such coefficient in  $F_n$  is *completely determined by the previous codes*. This conclusion finds an application of great practical use in the derivation of expansions as a temperature or  $u$ -grouping.

**5. Explicit illustration of latent symmetry and the application of the principle of partial balance**

We give in table 1 an analysis of the codes corresponding to the first five powers of  $u$  on the triangular lattice in  $F_{1,2}$ . The first three powers illustrate the patent symmetry corresponding to graphs of  $\kappa = 1$  only; the first entry belonging to the even regime, the second to the odd and so on alternately. All the codes in the second column

**Table 1.** Analysis of leading coefficients in  $F_{12}$ .

Power $u$ on triangular lattice	Corresponding codes (total)	$\kappa = 1$ one component no holes	$\kappa = 0$ one component one hole	$\kappa = 2$ two components no holes
12	1(19, 9, 3, 7)	1		
	1(18, 6, 6, 6)	1		
13	15(20, 11, 2, 7)	15		
	87(19, 8, 5, 6)	87		
	15(18, 5, 8, 5)	15		
14	6(21, 13, 1, 7)	6		
	345(20, 10, 4, 6)	345		
	345(19, 7, 7, 5)	345		
	6(18, 4, 10, 4)	6		
15	74(21, 12, 3, 6)	386		- 312
	2079(20, 9, 6, 5)	2391		- 312
	386(19, 6, 9, 4)	386		
16	- 732(22, 14, 2, 6)	78		- 810
	- 201(21, 11, 5, 5)	5160	9	- 5370
	4413(20, 8, 8, 4)	5160	63	- 810
	87(19, 5, 11, 3)	78	9	

corresponding to  $u^{12}$ ,  $u^{13}$  and  $u^{14}$  are determined by the results of the previous section. The *secondary* code (19, 9, 3, 7) is found by an application of the principle of partial balance and the primary code (18, 6, 6, 6) follows by symmetry. Once this latter code is known the principle of partial balance enables the secondary codes (20, 11, 2, 7) and (19, 8, 5, 6) to be found and the primary code (18, 5, 8, 5) follows by symmetry. The four codes in  $u^{14}$  can be filled in by repeating the process.

The powers of  $u^{15}$  and  $u^{16}$  illustrate the latent symmetry property. The symmetry of the codes corresponding to  $\kappa = 1$  is no longer visible in the total because of the contribution from codes corresponding to  $\kappa = 2$  (in  $u^{15}$ ) and later  $\kappa = 2$  and  $\kappa = 0$  (in  $u^{16}$ ). At this stage the coefficient is not determined by the previous codes but it is sufficient to provide all but one column of the analysis. Any column with  $\kappa \geq 1$  can then be completed by balance and symmetry. It is convenient at this stage to provide the columns  $\kappa = 2$  and  $\kappa = 0$ ; the full coefficients can then be completed. The process can then be repeated for the leading terms in  $F_{13}$ ,  $F_{14}$ ... until the powers of  $u$  through  $u^{16}$  terminate. In this way the ferromagnetic polynomials  $\psi_{12}$  through  $\psi_{16}$  are readily completed; since the coefficients are obtained by way of the codes the corresponding polynomials for the honeycomb lattice, both ferromagnetic and antiferromagnetic, are completed at the same time. To obtain further polynomials the provision of partial data corresponding to restricted values of  $\kappa$  needs investigation and we consider this in a companion paper (VIII).

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